Modal versus nonmodal linear stability analysis of river dunes

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When a free surface turbulent shear flow interacts with a deformable cohesionless sediment bottom, dune patterns can arise and perform a complex time evolution. These bed forms are very widespread in fluvial environments and have catalyzed an intense research activity of both an applicative and theoretical nature. This work investigates the non-normality of the linearized mathematical operator which rules the initial value problem, in order to detect possible transient growths and make a comparison with the picture described by the outcome of the eigenvalue problem. The dune dynamics have resulted to be heavily non-normal in large regions of the parameter space and to be able to develop important transient growths, even for asymptotically stable wavenumbers. The effects on progressive wavelength elongation that has been observed in some experiments are discussed, and an explanation, based on purely linearly mechanisms, is proposed and compared with some experimental data. A discussion about the saturation length concept in the sediment transport modeling is also proposed. © 2011 American Institute of Physics. [doi:10.1063/1.3644673]

I. INTRODUCTION

River dunes are a kind of bed form that belongs to the multifaceted variety of fluvial patterns, in which a loose boundary (the bed) interacts with a shear flow (the water stream) and a free surface (the air-water interface). A multitude of open questions makes river dunes one of the most studied phenomena in fluvial morphodynamics and, at the same time, a topic that still stimulates an exciting debate concerning several issues: the fascinating aspect of generation mechanisms,\(^1\) features of the temporal evolution,\(^2,3\) the occurrence of flow separation,\(^4\) the role of turbulent bursting phenomena,\(^5\) tridimensionality,\(^6\) suspended load,\(^7\) nonlinearities,\(^8\) ripple-dune transition,\(^9\) the influence on flow resistance and sediment transport,\(^9\) and the consequences on river management and navigation.

Unlike other forms of river patterns (e.g., ripples, bars, meanders), dune dynamics requires a thorough description of both the flow field structure and the free-surface. Dunes in fact scale with the depth of the water current and are, thus, closely related to the two dimensional (2D) (or three dimensional (3D)) features of the fluid dynamics over the whole stream (from the bed to surface), which in turn depend on the kinematic and dynamic behavior of the air-water interface. For these reasons, neither shallow water approximation, where vertical velocity is neglected, nor boundary layer approximation, where the role of the upper boundary disappears, ensures an appropriate modeling.

Since the seminal work by Kennedy,\(^10\) it is nowadays well-established that dune bed forms are the result of an instability process of the bed, where the sediment transport dynamics closely interacts with the complicated structure of a turbulent open channel flow. In order to disclose the fluid dynamic mechanisms, whereby sediment transport lags with respect to the bed topography and triggers instability, several models have been proposed, ranging from mixing layer analogy,\(^11\) to potential-flow,\(^12\) and rotational-flow models.\(^7,13–18\) The latter approach offers the most advanced formulation of the flow field and has been shown to be successful in providing a good prediction of the wavelength selection. Most of these works are based on the linear stability analysis (LSA) of the fluid momentum and sediment conservation equations (flanked by suitable closure relationships).

Notwithstanding this, when theory is compared to experiments on bed forms which develop from flat bed conditions, one of the most common criticisms raised concerning LSA is that the theoretical prediction fits data well in the long-term, when the dynamics should be driven by nonlinear mechanisms, but it fails at the initial times when, due to the small value of the perturbations, a linearized approach would be correct. Indeed, the LSA results invariably overestimate the wavelength selection at the beginning of experiments. This failure in the transient period is usually ascribed to the role of nonlinearities,\(^2,18\) but such a justification is in contrast with the good performance in the long-term. The aim of the present paper is to resolve the latter contradiction by investigating the non-normal (or nonmodal) character of the morphodynamic problem and to show that LSA is actually able to describe the initial times of instability and the successive wave elongation that has been observed in experiments.

The goal of the modal approach is to establish the asymptotic temporal fate of the disturbances, since focus is on the least stable eigenvalue. In this way, one evaluates whether disturbances tend to zero or infinity when time tends to infinity and, accordingly, classifies the basic state as stable or unstable. However, no information is gained on the behavior at finite times. Points in the parameter space are classified stable, regardless of whether the disturbances tend asymptotically to zero in a monotonic way or transient growths occur at finite times. The mathematical reason for these two different types of behavior is the degree of nonorthogonality of the eigenfunctions of the differential operator that

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describes the disturbance dynamics.\textsuperscript{19} Although the long-term asymptotic fate of the system is correctly driven by the least stable mode, a (modal) eigenvalue analysis is not a good descriptor of the transient behavior of the fluctuations when the eigenfunctions do not form an orthogonal set; in other words, they are non-normal.

A nonmodal analysis takes into account the non-normality of eigenfunctions and the proprieties of the whole spectrum of eigenvalues (and not only the least stable ones), through the reconstruction of the solution of initial value problems (IVPs) and the detection of transient growths. Since the hypothesis of an exponential dependence on time of the solutions is relaxed, the nonmodal approach is able to describe the initial stage of the instability and naturally englobes the results of the modal analysis as its asymptotic limit. We point out that the expression “nonmodal” analysis should not be confused with the absence of a modal basis—the use of normal modes, as a spatial basis, is in fact retained—it is instead just considered as a synonym of “non-normal” analysis.\textsuperscript{20}

Modal analysis constitutes a very important tool in theoretical research and has proved to be successful in fluid mechanics in the last century.\textsuperscript{21,22} Investigations on non-normality have instead attracted a great deal of attention in physics over the last two decades (e.g., see Ref. 23), where the domain of application includes, in particular, in fluid mechanics.\textsuperscript{20,23–26} It is now clear that most of the hydrodynamic operators that arise from a LSA of a shear flow are non normal and can lead to transient growths through a completely linear mechanism.

This approach has never been accomplished in the morphodynamic context, the only exception being one of our previous works on long-waves in de Saint Venant-Exner equations.\textsuperscript{27} One reason for this lies in the praxis of decoupling sediment dynamics from fluid dynamics by retaining temporal derivatives in only the former, i.e., the so-called steady-state approximation (an exception is given in Ref. 17). This approximation is suitable for the case of a modal analysis, but it precludes the investigation of transient growths.

The nonmodal approach to the linear stability analysis also gives us the opportunity to discuss some fundamental aspects of the sediment transport modeling. In the last decade, different improvements have in fact been proposed but not compared.\textsuperscript{18,28} In order to make our study on dune non-normality more robust, we will pay attention to analyze the impact of such aspects on the stability analysis.

II. MATHEMATICAL MODEL OF THE PHYSICAL PROCESSES

A derivation of the fourth-order differential operator of the eigenvalue problem (EP) through a vertical velocity formalism is here provided. Let us consider a turbulent open-channel flow which interacts with a bed composed of cohesionless, homogeneous sand. Reference system \{x, y, z\} denotes the right-handed Cartesian frame, where x is tangent to the base plane and parallel to the direction of the maximum slope and z is orthogonal to the base plane and points upwards (see Figure 1). The base plane forms an angle \(\beta\) with the horizontal. The fluid is incompressible and the suspended sediment load is assumed to be negligible. The dynamics of the mobile bed is, therefore, only regulated by sediment bed transport. The interface between the bed and the flow is represented by the surface, \(z = \Theta(x, y, t)\), while the upper interface between the water and air, namely, the free surface, is given by \(z = H(x, y, t)\).

The governing equations for the hydrodynamics are the Reynolds equations for longitudinal, vertical, and spanwise momentum conservation, along with the continuity equation. If the variables are made dimensionless using the friction velocity, \(u_*\), and the mean depth of the flow, \(D_0\), the model reads

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P + \nabla \cdot \mathbf{T} + \mathbf{f} = 0, \quad \nabla \cdot \mathbf{u} = 0, \tag{1}
\]

where \(\mathbf{u} = \{U, V, W\}^T\) is the velocity vector, \(P\) is the pressure, \(\mathbf{T}\) is the deviatoric stress tensor, \(\mathbf{f} = (-1, 0, S^{-1})^T\), and \(S = \tan \beta\) is the longitudinal mean bed slope (see Figure 1). In order to elucidate the basic features of the EP and IVP solutions and to make a simple comparison with the experimental data, three-dimensional perturbations are not considered hereafter, but the transient behavior of purely cylindrical dunes is focused on. Accordingly, Squire’s equation for vorticity conservation is not used and y-derivatives are neglected. Although three-dimensional dunes naturally emerge in most cases, the very early development of the incipient perturbations is preferentially two-dimensional.\textsuperscript{6} A vast amount of laboratory literature has only focused on 2D structures, usually forced by the adoption of narrow flumes.\textsuperscript{20}

We now introduce the mapping \(\zeta = 2(z - \Theta)/D - 1\), where \(D(x, t)\) is the local value of the stream depth. In this way, the domain \(z \in [\Theta(x, t), \Theta(x, t) + D(x, t)]\) transforms into \(\zeta \in [-1, 1]\). Such a change simplifies the boundary conditions and is suitable for the spectral numerical approach adopted here. We also consider a Boussinesque-type closure of turbulence through the relationship \(T = -2\nu_T S\), where \(S\) is the tensor of deformation rate, whereas the eddy viscosity \(\nu_T\) is computed through a mixing-length model.\textsuperscript{30} Namely, \(\nu_T = fD^2|2S \cdot S|^{1/2}\). Here, we set a mixing length dependence on \(\zeta\), which assures both a good agreement with empirical data in open-channel flows\textsuperscript{31} and analytical ease, namely,

\[
I = \frac{\kappa \sqrt{2}}{2\sqrt{2}} (\zeta + \zeta_r + 1), \tag{2}
\]

where \(\kappa \simeq 0.41\) is the von Karman constant and \(\zeta_r\) is the roughness displacement. In the following, we will adopt (one of) the usual hydraulic empirical formula, \(\zeta_r = d_s/12\), \(d_s\)
being the relative roughness, i.e., the ratio between the mean sediment diameter, $d_{so}$, size and the mean depth, $D_0$.

In order to develop an LSA, a normal mode perturbation is applied to Eq. (1) in the classical form

$$\{U, W, P, \Theta, D\} = \{U_0(\zeta), 0, P_0(\zeta), 0, 1\}
+ \epsilon \{\bar{u}(\zeta, t), \bar{w}(\zeta, t), \bar{p}(\zeta, t), \bar{h}(t), \bar{D}(t)\}e^{i\omega t},$$

(3)

(plus complex conjugate), where $\epsilon \ll 1$ and $\omega$ is the real longitudinal wavenumber. After linearizing, at order $O(1)$, the leading term is obtained, namely, the well-known logarithmic profile for the basic state velocity and (linear) hydrostatic pressure distribution. At order $O(\epsilon)$, the pressure mode, $\bar{p}$, can be eliminated through cross derivation of the longitudinal and vertical components of Eq. (1). Furthermore, using the continuity equation, which reads $i\bar{u} \bar{u}_t = -2\bar{w} \bar{h}_t$, we reach a fourth-order ordinary differential equation (ODE) equation for $\bar{w}$, in a similar fashion to the Orr-Sommerfeld (OS) equation for stability analysis of a laminar shear flow.22, 32

$$a_i \Delta_i \bar{w} + a_3 \bar{h} + a_\tau (x^2 - D^2) \frac{\partial \bar{w}}{\partial t} = 0, \quad i = \{0, 4\},$$

(4)

where $\Delta_i = \partial / \partial z_i$ (the Einstein summation convention has been adopted), $\bar{\Delta}_i = \bar{h} + \bar{H}$, and $a_i$ are $z_i$-dependent functions (see Appendix A in supplementary material). Equation (4) can be formally written in operator form as

$$\frac{d \bar{q}}{dt} = \mathbf{L} \bar{q},$$

(5)

where $\bar{q} = \{\bar{w}(\zeta, t), \bar{h}(t)\}^T$, $\mathbf{L} = \mathbf{M}^{-1} \mathbf{S}$

$$\mathbf{M} = \{a_\tau (\Delta_i^2 - x_i^2), 0, 0\}, \quad \mathbf{S} = \{a_i \Delta_i, a_5, a_6\},$$

(6)

which leads to the formal solution $\bar{q} = \bar{q}_0 \exp(\mathbf{L} t)$, being $\bar{q}_0$ the initial condition.

Finally, we assume an exponential dependence on time, in this way addressing an EP, namely, $\bar{q} = q_0 \exp(-i\omega t)$, where $\lambda$ is the complex eigenvalue and $q = \{w(\zeta), \eta, h\}^T$. With these latter changes, the eigenvalue problem for the stability analysis is written in the operator form

$$\mathbf{S} \bar{q} = \lambda(-i\mathbf{M}) \bar{q},$$

(7)

where $\mathbf{S}$ and $\mathbf{M}$ are usually called the stiffness and mass operators.

In order to formally complete the mathematical problem, the above model requires that the eigenvalue, $\lambda$, and the eigentriple, $\{w(\zeta), \eta, h\}^T$, are found after six boundary conditions have been added:

$$w + b_1 h = \lambda(-i)h,$$

(8a)

$$b_2 + b_3 D^2 + b_4 \eta + b_5 h = 0, \quad (\zeta = 1),$$

(8b)

$$b_6 + b_7 \Delta + b_8 D^2)w + b_9 \eta + b_10 h = \lambda(-2i)Dw, \quad (\zeta = 1),$$

(9)

$$2i Dw + b_11 \eta = 0, \quad (\zeta = -1),$$

(10a)

$$(b_12 + b_13 D^2)w + b_14 \eta + b_15 h = \lambda(i)\eta, \quad (\zeta = \zeta_b).$$

(11)

The above equations describe: (1) at the free surface ($\zeta = 1$), the kinematic condition (8a), the tangential component— with respect to the free surface—of the dynamic condition (8b), and the normal component of the dynamic condition (9); (2) at the bottom ($\zeta = -1$), the kinematic condition (10a) and the no-slip condition (10b); (3) at the bed-flow interface ($\zeta = \zeta_b$), the conservation of mass for the sediment phase, i.e., the so-called Exner equation (11).

The algebraic derivation of the coefficients $b_i$ is standard, but cumbersome; their expressions are reported in the supplementary material.36 We only remark the physical contents of the Exner equation (11) which is the key element of bed form instabilities and it is functional to the analysis provided in Sec. IV. The Exner equation calls for the horizontal gradient of the volumetric bedload discharge which, in the case, the horizontal bed, can be accounted for using one of the many empirical formulas reported in literature.33 Most of these formulas, if posed in a suitable dimensionless form, share the structure $\Phi(\theta, \theta_b) = \sigma_0 (\theta - \theta_b)^{\gamma}$, where $\Phi$ is the dimensionless bedload discharge, $\theta = \tau_c (\langle d \rangle R)^{-1}$ is the Shields stress computed at the bed, $\theta_b$ is its critical threshold for the incipient sediment mobilization over a flat horizontal bed, $\tau_c$ is the modulus of the dimensional shear stress at $\zeta = -1$, $\gamma$ is the fluid specific weight, and $R$ is the relative density of the sediment. We have adopted the formula by Fernandez Luque and van Beek34 ($\sigma_0 = 5.7$, $\sigma_1 = 1.5$), but we have also verified that the more common formula by Meyer-Peter and Muller35 does not lead to any significant change in the results.

Three papers on the modeling of dune dynamics have improved the above mentioned structure of the bedload, stemming from: (1) the Danish school by Fredsoe,13 (2) the Italian school by Colombini and coworkers,7,16,17 and (3) the French school by Andreotti and coworkers.18,28 Apart from (1), which is the oldest and most consolidated, the mechanisms elucidated by (2) and (3) were almost ignored in the past literature.

(1) Fredsoe13 introduced a correction in the Shields stress threshold when the bed is far from being horizontal, in order to consider the role of gravity, which opposes uphill motion and favors downhill motion. The modified critical Shields stress reads $\theta_c = \theta_b [1 - (S - \Theta_c) / \mu]$, where $\mu$ is the Coulomb coefficients. This correction has a stabilizing effect on the model and it is called the gravitational correction hereafter.

(2) Colombini16 claimed that the Shields stress in the bed-load sediment transport formula should be “evaluated at the interface between the flowing fluid and the (thin) saltation layer where grains are involved in bedload transport processes” and not at the bottom. The term $\theta_c$ should, therefore, be replaced by $\theta_b = \tau_b (\langle d \rangle R)^{-1}$, where $\tau_b$ is the modulus of the shear stress evaluated at $\zeta = \zeta_b$. The dimensionless height $\zeta_b$ of the saltation layer can be estimated on the basis of some empirical formula, such
as the one by Sekine and Kikkawa.\textsuperscript{36} Colombini has shown that such a modification, henceforth referred to as the \textit{saltation} correction, is crucial to forecast anti-dune instability, even in the absence of a suspended load. With the gravitational and saltation corrections, the \textit{x}-dependent bedload discharge is formally represented by the function $\Phi(\theta_x, \theta_y)$.

(3) The “flow saturation” concept introduced by Parker\textsuperscript{37} and mainly adopted by Andreotti and co-workers\textsuperscript{18,28,38,39} claims that, in the presence of a perturbed bottom, the local bedload volumetric discharge is far from the equilibrium condition foreseen by $\Phi(\theta_x, \theta_y)$, because of the inertial response of the sediment particles to the continuous changes in shear stress. Accordingly, the actual value of the varying bedload discharge $q$ is modeled through a first-order relaxation equation of the form $L_{sat} \frac{dq}{dt} = \Phi - q$, where $L_{sat}$ is the characteristic length of saturation. Unfortunately, no consistent formulas are known for $L_{sat}$ in a water stream, thus it usually requires \textit{a-posteriori} calibration. If the relaxation equation is Fourier-transformed, in accordance with (3), the ratio between the local and equilibrium bedloads is equal to the quantity $r = (1 + i\alpha L_{sat})^{-1}$. In order to incorporate such a correction into the above mathematical framework, it is sufficient to multiply $\Phi$ by $r$. If $L_{sat} = 0$, the so-called \textit{inertial} correction is neglected.

We finally recall that a positive (negative) imaginary part of eigenvalue $\lambda$ implies asymptotic instability (stability) of the infinitesimal perturbations and $c = \text{Re}(\lambda)/\alpha$ is the phase velocity. It is also straightforward to show, through the use of the Keulegan equation\textsuperscript{33} and the basic state solutions, that the above mathematical problem contains three independent parameters: the Froude number, $F = U_0/(gD_0)^{1/2}$ ($U_0$ is the bulk velocity and $g$ is the gravity acceleration), the Chezy coefficient, $C = U_0/\sqrt{g}$, and the wavenumber, $\alpha$.  

**III. METHODS**

**A. Spectral solution of the eigenvalue problem**

In order to recast the eigenvalue problems (7)-(11) in the algebraic form $A \mathbf{x} = \lambda B \mathbf{x}$, we discretize the problem using a spectral Galerkin technique with numerical integration (GNI), which prevents the onset of spurious eigenvalues (see Refs. 40, 41). This technique consists of three main steps. First, a modal representation of the solution is adopted, where the eigenfunction $w(\xi)$ is expanded in the (truncated) spectral form $w = \sum_{k=1}^{N} \phi_k(\xi)$, with $k = -3, \ldots, N = 3$, where $\phi_k(\xi)$ are called trial functions and $\phi_k$ are the unknown complex coefficients. Second, Eqs. (7)-(11) are multiplied by a set of \textit{test} functions and integrated over the domain $[-1,1]$. The repeated use of integration by part allows the fourth and third derivatives to be reduced to second-order derivatives; in addition, the boundary condition (9) is incorporated in the mass and stiffness operators in the so called \textit{weak} form,\textsuperscript{41} through the boundary term that arise from the integration by parts. Finally, the resulting generalized algebraic eigenvalue problem is solved, the unknowns being represented by the vector $\mathbf{x} = \{\phi_k, \eta, \xi\}_T$ and the eigenvalue $\lambda$. The formal mathematical derivation of the aforementioned three steps is given in Appendix B of the supplementary material.\textsuperscript{36} The resulting algebraic problem is easily solved by means of the QZ-algorithm. Next, once such a system has been solved, we eventually obtain the numerical values of the coefficients $\{\omega_{-2}, \eta, \xi\}$, by solving systems (A1)-(A3) of the Appendix and an extended vector $\mathbf{x} = \{\phi_k\}$, with $k = (-3,\ldots,N-3)$, which contains all the coefficients $\phi_k$ (including $\omega_{-2}$), is built. Vector $\mathbf{x}$ is fundamental for the analysis that is developed in the following section.

**B. Analysis of the transient behavior**

A standard way of facing the IVPs (5) actually precludes passing to an eigenvalue problem like Eq. (7), since it can formally be solved using a spectral discretization of operator $L$, which eventually leads to a standard ODE. This is the so-called \textit{method of lines}, which require to set an initial condition, usually chosen to be physically realistic.\textsuperscript{42} If one is interested in detecting the \textit{optimal condition} that leads to the largest response, a suitable cost function has to be maximized with the aid of some penalty terms.\textsuperscript{43} Although such a methodology allows an \textit{a-priori} characterization of the shape of the perturbation to be made, it usually demands a large computational effort and it is weakly stable for very non-normal operators (as in the present case).\textsuperscript{19}

In this work, we have opted for another technique which uses the \textit{singular value decomposition} (s.v.d)\textsuperscript{44} and provides the optimal condition and the corresponding response in a simpler way than the method of lines.\textsuperscript{25,45} This method is based on an eigenfunction decomposition and it maintains a sufficient computational robustness, even for the case of high levels of non-normality. This is the reason why we have pursued on the EP in Sec. II.

The study of transient behavior and non-normality requires the choice of a norm that describes the growth evolution. It is physically relevant to refer to the evolution of the total energy that is associated to the fluctuations.\textsuperscript{30,25,46} In the present problem, the total energy is represented by the sum of three terms: the kinetic energy of the flow field, $E_k$, the potential energy of the free surface, $E_{\eta}$, and the potential energy of the bed, $E_{\xi}$. In particular, the presence of the last form of energy is one of the main novelties of the present work. In this way, we extend what we previously developed in the context of 1D long-wave river morphodynamics\textsuperscript{57} and non-Newtonian viscous stratified layers.\textsuperscript{57} The computation of the bed-induced potential energy is indeed a key element for the investigation of the transient behavior of any morphological instability, and it is here introduced in order to study dune instability induced by two-dimensional flows.

If capillary forces are neglected and the null potential is set on the undisturbed surface, with the aid of the continuity equation, the density of the total energy in the wavenumber space can be written as

$$E = \frac{1}{4\pi^2} \int_{-1}^{1} \left( 4|Dw|^2 + z^2 |w|^2 \right) d\zeta + \frac{|\eta|^2}{2F^2} + r \frac{|\eta|^2}{2F^2}, \quad (12)$$

where $r = (1 - p)R$ and $p$ are the sediment porosity.
Let us introduce the solution technique of the IVPs through an eigenfunction expansion, extending some previous treatments to the present morphodynamic problem. Once the (complete) space of eigenfunctions \( S^N = \text{span} \{ q_1, q_2, \ldots, q_N \} \) and the eigenvalue diagonal matrix \( A = \text{diag} \{ \lambda_1, \lambda_2, \ldots, \lambda_N \} \) are known from the EPs (7)-(11), one is able to write the solution \( \hat{q}(t) \) of the IVPs in terms of expansion of \( S^N \). It follows that we seek the time-dependent coefficient \( \{ \kappa_1(t), \kappa_2(t), \ldots, \kappa_N(t) \}^T = \kappa \), such that \( \hat{q}(t) = \kappa(t)q \) with \( i = 1, n \) and \( 1 < n \ll N \). It is important to notice that the IVP (5) becomes \( d\kappa/\!\!\!dt = -i\kappa \). The corresponding representation, in terms of the three components of the IVP solution \( \{ \ddot{w}, \ddot{\eta}, \dot{h} \} \), is

\[
\dot{\kappa}(\zeta, t) = w(\zeta) \cdot \kappa(t), \quad \dot{\eta}(t) = \eta \cdot \kappa(t) \quad \dot{h}(t) = h \cdot \kappa(t),
\]

where \( w = \{ w_1, w_2, \ldots, w_n \} \), \( \eta = \{ \eta_1, \eta_2, \ldots, \eta_n \} \), and \( h = \{ h_1, h_2, \ldots, h_n \} \). The bounds on the truncation value, \( n \), stem from the fact that the number of eigenfunctions used in the expansion has to be sufficiently high to be able to capture the non-normality of the spectrum correctly, without being affected by the loss of accuracy that is present at the highest modes of the computed spectrum.

By using Eq. (13), the kinetic energy is written in terms of the eigenfunction expansion,

\[
E_k = \frac{1}{4\pi} \int_{-1}^{1} \left[ 4(Dw \cdot \kappa)^2 + \kappa^2 \right] d\zeta,
\]

where \( \kappa^2 = \frac{1}{4\pi} \int_{-1}^{1} \left[ 4(Dw \cdot \kappa)^2 + \kappa^2 \right] d\zeta \).

A straightforward computation of the symmetric matrix \( W \) is obtained with the aid of the orthogonal proprieties of the Legendre polynomials, but its derivation is algebraically tedious and has not been shown here for the sake of space.

A similar procedure can be extended to both forms of potential energy; thus, if we define \( a = F^{-2}/2 \), we obtain

\[
E = \kappa^T \left( A + r h^T h \right) \kappa + \kappa^T M \kappa = (F \kappa)^H (F \kappa) = \left\| F \kappa \right\|_E^2 = \left\| \kappa \right\|_E^2.
\]

The above sequence of equalities requires some specification. First, since \( M \) is a positive definite symmetric matrix, it has been decomposed into the form \( M = F^H F \). It is straightforward to show that \( F \) is obtainable from the s.v.d. \( M = USV^H \), according to \( F = S^{1/2} U \), where \( S \) is the diagonal matrix of the singular values and \( U \) is the related left unitary matrix. Second, notation \( \left\| \alpha \right\|_E \) refers to the well-known Euclidean 2-norm of a generic vector \( \alpha \). Third, the last identity in Eq. (16) introduces the definition of a new energy norm applied to the vector of the coefficients \( \kappa \). We finally recall that the corresponding energy norm of a generic matrix \( A \) is \( \left\| A \right\|_E = \left\| FA F^{-1} \right\|_E \) (e.g., Ref. 22).

At this point, we can compute the so-called growth function and numerical abscissa. These quantities require a matrix operator that derives from the original mathematical problem and a norm definition. From the above considerations, the matrix operator results to be \( \Omega = FA F^{-1} \) and the norm is the energy norm defined in Eq. (16).

The growth function, \( \hat{G}(t) \), is defined as the upper envelope of the normalized total energy density, \( G(t) = E(t)/E(0) \), which reads (see also Ref. 19)

\[
\hat{G}(t) = \max_{q_0} \left\{ \frac{\| \tilde{q}(t) \|_E^2}{\| q_0 \|_E^2} = \max_{\kappa_0} \frac{\| \kappa(t) \|_E^2}{\| \kappa_0 \|_E^2} = \| \exp(-it\Omega) \|_E = \left\| \text{exp}(-it\Omega)F^{-1} \right\|_E^2 \right\}.
\]

It is evident that the occurrence of \( \hat{G}(t) > 1 \) for any \( t > 0 \), indicates transient growths, regardless of the asymptotic behavior of the system.

The supremum of the real part of the eigenvalues describes the asymptotic fate, while the supremum of thereal part of the numerical range (i.e., \( W \equiv \{ x^H \Omega x : x \in C^N, \| x \| = 1 \} \)) is the so-called numerical abscissa, \( n_s(\Omega) \), which marks the behavior at the beginning time. In fact, according to the Hille-Yosida and Lumer-Phillips theorems, \( 16 \) it follows that

\[
\frac{d}{dt} \left\| \hat{G} \right\|_{t=0} = n_s(\Omega) = \frac{1}{2} \left[ \Omega + \Omega^* \right],
\]

where subscript \( t \) refers to the largest eigenvalue.

IV. RESULTS

A. Modal analysis of asymptotic behavior

We have designed four runs with a fixed Chezy coefficient \( C = 20 \), the same as the one analyzed in Refs. 16 and 18 \) and a variable Froude number and wavenumber \( F = [0.2–1.2], x = [10^{-3}–10^{-2}] \), in order to also investigate ripple formation region, as suggested by Fourriere et al. 18

Run \( R_g \) only meets the Fredsøe prescription, run \( R_b \) uses the same model as Colombini, 16 where the gravitational and
the saltation corrections are both implemented, but not the inertial correction ($L_{sat} = 0$). Run R$_a$ only follows the inertial correction ($L_{sat} = 8d_i$). It should be notice that this run is identical to the simulations made by Fourriere et al. $^{18}$ Run R$_d$ implements all the three corrections and, thus, accounts for all the modeling aspects. Apart from the modeling aspects, all the simulations share the same numerical method of solution of the eigenvalue problem, as illustrated in Sec. III A. The results on the growth rate of the least stable eigenvalue and the associated phase velocity are reported in the semi-logarithmic plots of Figure 2.

A first important finding emerges from all the simulations; regardless of the modeling choices, two regions of instability are always present, namely, at low and high wavenumbers, $\alpha = O(10^{-1})$ and $\alpha = O(10^2)$, respectively. The former instability is characteristic of dune patterns, while the latter could be associated to ripple formation. Furthermore, the second instability region exhibits a much larger growth factor. Such an independence of the emergence of the two instability regions on the modeling aspects is quite astonishing and seems to confute the statement by Fourriere et al. $^{18}$ where the inertial correction was postulated to be a necessary and sufficient condition to model ripple instability.

Other important aspects emerge from the results of the stability analysis. First, a third type of instability, with a negative celerity, has been recognized at high Froude numbers in runs R$_b$ and R$_d$. This means that antidunes are correctly modeled only if the saltation correction is considered, as pointed out by Colombini $^{16}$ otherwise there is an anomalous merging between the two zones (see R$_a$ and R$_c$). On the other hand, the saltation correction leads to the inconvenience of a negative celerity in the ripple region. Second, the gravitational correction weakly reduces the upper bound in the frequency of the dune instability region. Third, if the indications of Fourriere et al. $^{18}$ are followed (i.e., just inertial correction, see run R$_c$), the phase velocity in the ripple instability is positive, but unrealistically high, $c \sim 1$, which, in dimensional terms, leads bed form celerity comparable to the friction velocity.

At this point, one could wonder why a positive growth rate invariably emerges at high frequencies in Fig. 2. We have repeated the calculations of Fig. 2 with an increased number of modes, $N$, for the spectral discretization, and the results appeared to be the same. Thus, no numerical instability affects the results. An answer can be instead given by recalling that the present model only considers rough flow regime, and the roughness height, $z_r$, is the only short-scale present in the mathematical framework which is able to interact with the highest wavenumbers. It can be verified that an artificial modification of the value of $z_r$ in the computation of the morphodynamic coefficients, affects the ripple instability region; if $z_r$ is forced to increase, such a region emerges at smaller wavenumbers. This point has been recently elucidated by a paper appeared during the review process of the present work, $^{18}$ which already discussed the occurrence of spurious instability at high wavenumbers and emphasized that only dunes and antidunes are present in the rough regime, whereas ripples belong to transitional and smooth flow regimes. To sum up, the present model should be limited to the study of dune instability (say $\alpha < 10$). Under this value, the adoption of just the gravitational and saltation corrections assures a correct modeling of dune dynamics; otherwise, the model triggers a fictitious instability, which must not be confused with the detection of physical phenomena.

### B. Nonmodal analysis of transient behavior

Let us begin the analysis of the transient behavior by focusing on the growth function $\tilde{G}(t)$. We have reported, in Figure 3, the behavior of $\tilde{G}(t)$ for two different parameter sets, which, from an asymptotic point of view, correspond to a stable case (Fig. 3(a)) and an unstable antidune forming case (Fig. 3(b)). Accordingly, in the former (latter) case, the least stable eigenvalue has a negative (positive) imaginary part. In particular, Figure 3(b) shows the outcome of the growth function, either including (solid line) or excluding (dashed line) the least stable eigenvalue in the computations of Eq. (17).

Regardless of the long-term behavior, the growth function discloses a transitory growth of the total energy of the perturbation, that is, up to $10^3$ times its initial value, due to the non-normality of the operator. Furthermore, the remarkable oscillatory structure of the curves is related to the time-frequency of the least stable mode, that is, the real part of the eigenvalue with the highest imaginary part, as already
growth of the least stable eigenvalue. In Sec. dominant) growth, followed by a decrease and then a weak unstable case. The solid line in fact shows a beginning (pre-
sient behavior can be significant even in the asymptotically 
means that (1) the first rise in the instability is mainly driven 
attractability of wavelength versus time. 

It is also instructive to observe the role of the (positive) least stable eigenvalue, through an inspection of Fig. 3(b). The dashed and the continuous lines just begin to separate 
least stable eigenvalue, through an inspection of Fig. 3(b). (a) Input: 

The dashed and the continuous lines just begin to separate 
least stable eigenvalue, through an inspection of Fig. 3(b). (a) Input: 

pointed out for falling liquids, channel flows, and non-
Newtonian fluid layers. It is also instructive to observe the role of the (positive) least stable eigenvalue, through an inspection of Fig. 3(b). The dashed and the continuous lines just begin to separate appreciably in the mature part of the graph ($t > 15$). This means that (1) the first rise in the instability is mainly driven by the negative part of the spectrum ($\lambda_i, i > 2$) and (2) transient behavior can be significant even in the asymptotically unstable case. The solid line in fact shows a beginning (predominant) growth, followed by a decrease and then a weak long-term rise, which asymptotically tends to the exponential growth of the least stable eigenvalue. In Sec. V, we will recover this preliminary remark in order to show the linear superposition of wavelength versus time.

With the aim of facilitating the description of the transient behavior over a wide range of the parametric set, without any loss of generality, we focus on the root square of the growth function, $g = \hat{G}^{1/2}$, which we conventionally name the reduced growth function. Unlike the energy-leading quantity $G$, the function $g$ is usually preferred in the mathematical literature, since it is exactly the $L^2$-norm of the initial perturbation. For our purpose, both quantities can be used without distinction, and we have opted for the latter for just practical reasons: the numerical abscissa in fact results to be equal to the derivative of $g$ at $t = 0$ (see Eq. 19). Furthermore, three descriptors of $g(t)$ behavior can be computed in the following way:

$$E_g = \int_0^{T_{lim}} g(t)dt, \quad \mu_g = \frac{E_g}{T_{lim}}, \quad T_g = \frac{1}{E_g} \int_0^{T_{lim}} t \cdot g(t)dt,$$

where $T_{lim}$ is such that $g(t) < 1$ for $t > T_{lim}$, that is, the total time of the growth period. Being $E_g$ a (dimensionless) cumulated energy of perturbations during the transient growth, $\mu_g$ its mean value, and $T_g$ a characteristic timescale of the growth period, an overall analysis of these three quantities can provide an indication on the persistency and magnitude of the transient phenomena. Large values of $E_g$ and $T_g$ along with small mean values, $\mu_g$, indicate a low but persistent transient activity. On the contrary, high mean values and small $T_g$ identify ephemeral events. We observe that the evaluation of $T_{lim}$ is nonsense in the unstable case (i.e., Fig. 3(b), solid line); therefore, any positive eigenvalue will be disregarded in the computation of the reduced growth function. In Figs. 4-6. In this way, we will be able to evaluate the descriptors (20) even in the unstable regions and focus on the transitory dynamics before asymptotically stable modes become dominant.

Figure 4 reports the values of the quantities (20) for a fixed Chezy coefficient and a varying Froude number and wavenumber ($F = [0.2 - 2.5], \alpha = [10^{-2} - 2]$). The contour lines span a logarithmic scale and are represented in a color-scale for $E_g, \mu_g$, and $T_g$. Gray regions refer to asymptotically stable conditions which are delimited by the curve $\text{Im}(\lambda) = 0$.

FIG. 3. Growth function vs. time. (a) Input: $C = 15, \alpha = 0.07, F = 1.15, i = 1,...,50, T_{lim} = 100$; output: $E_g = 282, \mu_g = 2.82, T_g = 32$. (b) Input: $C = 15, \alpha = 0.4, F = 1.5, i = 1(2)...50$ for the solid (dashed) line, $T_{lim} = 50$; output: $E_g = 1440, \mu_g = 9.1, T_g = 104$ (solid line); $E_g = 150, \mu_g = 3.1, T_g = 16.6$ (dashed line).

FIG. 4. (Color) Overall description, in the $\alpha - F$ plane, of the asymptotic behavior through the quantity $\text{Im}(\lambda)$ (asymptotically unstable region, in gray) and the transient behavior through the quantities $E_g$ (a), $\mu_g$ (b), and $T_g$ (c) (with increasing values from blue to red). Colored regions are asymptotically stable. The white region at very low Froude numbers is characterized by the absence of sediment transport. The magnitudes of the contour levels are logarithmically spaced in the intervals $[10, 10^3], [1, 7.5], [1, 10^3]$ for $E_g, \mu_g$, and $T_g$, respectively. The details of the particular cases {P0, P1, P2, P3} are reported in Table I ($C = 15$).
The main result is quite evident: significant transient behavior is exhibited, regardless of the parametric conditions, as the high values of quantities $E_g$ and $\mu_g$ testify. Moreover, the higher the energy of the transient growth, the longer the timescale $T_g$ (compare Figs. 4(a) and 4(c)). A zone with high values of the growth function and long timescales, localized at $F \sim 1$, between the lower dune-instability and the upper antidune instability and relatively low wavenumbers, is remarkable. In this region, we can also observe a zone with high values of the growth function and long timescales, localized at $F \sim 1$. The increase in the non-normality for a hydraulically quasi-critical flow confirms a behavior that is well-known for 1D morphodynamic equations of shallow water rivers. The hyperbolic character of the de Saint Venant-Exner equations in fact suggests that one out of the three characteristic lines has a celerity which increases as $\sim (1 - F^2)^{-1}$. It follows that, at the critical conditions, the celerity of the morphodynamic response is not negligible and decoupling hydrodynamics from morphodynamics ($d/dr = 0$ in flow equations) is no longer allowed.

Another physically relevant region should be mentioned (marked with letter $R$ in Fig. 4(a)) that is located at very low wavenumbers with Froude numbers between 1.5 and 2.3. The transient activity is very high in this region, with very large values for $E_g$ and $T_g$. It is interesting to observe that this region does not correspond to a high mean values ($\mu_g$-plot). It follows that the high total energy of the disturbance, $E_g$, is caused by high values of the bound $T_{lim}$ in the integral (20) and that the perturbation is characterized by a relatively low maximum in the growth function, but persistence in $g > 1$ over time. This aspect is crucial for the time sequencing of the selected wavelengths. It is also important to recall that, for a fixed value of the Froude number, the coexistence of both asymptotically stable, but transiently unstable perturbations, with asymptotically unstable ones raises some interesting questions on the dynamics of (linear) interactions among different wavelets. This point is elucidated in Sec. V.

In order to further illustrate the features of the non-normality and the structure of the optimal disturbance, four particular points ($P_0, P_1, P_2, P_3$) have been selected in the $x-F$ plane. A first synthesis is provided in Table I, where the parametric coordinates ($x, F$), the corresponding values of the condition number, $K = \|V\|_E^{1/4} \|E^{-1}\|_F$, and the numerical abscissa, $n_a$, are reported. The condition number provides a synthetic measure of non-normality; the theory states that $K = 1$ ($K > 1$) for normal (non-normal) operators. Two different conditions are considered in Table I: the mobile bed condition, which is the main focus of the present work, and the flat bed condition, where the bed perturbation is forced to zero and only surface and hydrodynamic modes are free to evolve. In the following, for brevity, these two conditions will be abbreviated to MB and FB, respectively. We again recall that all the unstable eigenvalues have been removed from the computation. It appears that the non-normality of the FB problem is consistently higher than MB. This provides a first important novelty, with respect to the 1D shallow water system investigated by Camporeale and Ridolfi, where the de Saint Venant equations were shown to be almost normal, in contrast to de Saint Venant-Exner equations. The adjoint of a vertically variable 2D flow field has in fact increased the degree of freedom, from finite to infinite values, with a dramatic change in the non-normality.

Furthermore, even timescales should be considered, the MB problem usually evolves with timescales that are one or

<table>
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<th>$F$</th>
<th>$K$</th>
<th>$n_a$</th>
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<td>16</td>
<td>41.7</td>
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</table>

FIG. 5. Flow field of the optimal disturbances.

FIG. 6. (Color online) Transient growth versus time in $t \in [0, T_{lim}]$. Solid lines: reduced growth function, $g(t)$. Dashed curves: asymptotic prediction of the least stable mode. Panels (a)-(c) correspond to cases (P0)-(P2), respectively (mobile bed conditions).
two orders of magnitude longer than FB. The morphodynamic activity is in fact slower than the surface wavelets. Another interesting result is exhibited at the point $P_2$, where $K \sim 1$ for MB, thus testifying near-normal behavior. Such a result seems to suggest that no-transient behavior appear in the region of roll-wave instability, in agreement with the common experience.

The vertical structure of the 2D-flow field $\{\hat{u}_0, \hat{w}_0\}$ (obtained through Eq. (18)) for the optimal disturbances in the asymptotically stable, $P_1$, and unstable, $P_3$, cases is shown in Figure 5. The MB and FB conditions are compared for each case. The vertical coordinate is normalized with respect to the local depth. It can be observed that the structures of the optimal disturbance of the FB problem in $P_0$ and $P_3$ are substantially the same, since the surface modes are asymptotically stable in both situations. On the contrary, the two results concerning the MB problem are qualitatively different; $P_0$ has three characteristic zones, localized near the bottom, the middle, and the top of the domain, which display local divergence-convergence-divergence of the flow-field; this sequence is inverted in $P_3$. Another important role is played by the two boundaries of the domains, namely, the bed and the free surface. In FB, the no-slip condition on the bottom affects the upper flow field to a great extent, whereas the most important flow activity is concentrated near the free surface. In MB, the scenario is the opposite; although the no-slip condition remains satisfied, the maximum flow velocity of the optimal disturbance appears very close to the bed, in a region which interacts with the active layer of the bedload. Using Trefethen's terminology, both problems are triggered by “wall” optimal disturbances; the wall being the bed for MB and the free surface for FB. Finally, the flow field in FB shows more inversions in direction, particularly for the $\hat{u}_0$ component, which could be associated to a high dependence on the free surface activity. On the contrary, the flow field in MB undergoes a greater stabilization, because of the deformation of the domain from both sides.

A last result is that concerning the behavior of functions $g(t)$, which correspond to the same parametric conditions as $P_0$-$P_2$ (Fig. 6). It is possible to observe that the overall temporal evolution of the dynamics is much larger than $T_{\text{max}}$, i.e., the time corresponding to the maximum growth function; this latter cannot, therefore, be considered representative of the temporal evolution of the process. As already stated, the oscillatory structure of the growth function, related to the real part of the least stable eigenvalue, also indicates the existence of complex dynamics in the transient period where different disturbances alternate in a non-trivial manner. Finally, we emphasize the high values of $g(t)$ and its distance from the exponential asymptotic prediction, provided by the least stable eigenvalue, the only exception being case $P_2$, because of its low level of non-normality.

V. DISCUSSION

The theoretical analysis developed so far has an implication of applicative interest on the issue of pattern coarsening. Several authors have detected that, during bed form formation from a flat bed condition, the wavelength increases with time. Basically, many experiments have reported an evolution from a first set of short wavelets, resembling ripple patterns, towards longer dune-like bed forms. Such an evolution is achieved through the coalescence of two discrete bed forms that come together to form a single bed feature or through superimposition of the smaller dunes onto the stoss side of larger ones. This bed feature behavior is paralleled by an increase in the amplitude. A universal scaling of the time evolution from an initially flat bed has been proposed in the form

$$\frac{\alpha(t)}{\alpha_e} = \left(\frac{t}{t_e}\right)^{-\gamma},$$

where $\alpha_e$ is the (asymptotic) equilibrium wavenumber that is (conventionally) reached at the equilibrium time, $t = t_e$. The validity of Eq. (21) has been proved in the interval $0.01 < t/t_e < 1$. Coleman et al. improved the first estimation of the exponent $\gamma$ ($< 0.28$ reported by Nikora and Hicks with the empirical relationship $\gamma = 0.14d_0^{0.3} (gR/\nu)^{1/9}$, for the specific case of dunes ($\nu$ is the fluid kinematic viscosity).

Despite the amount of data available in the literature, Raudkivi claimed that no models are capable of “describing the development, shape, and transformation of the bed features shown by observations.” Furthermore, Coleman and Nikora, when reviewing the challenges in fluvial dune research, lamented that the wide spectrum of theories, even the rotational ones, still present “unresolved inconsistencies.” One of the most common explanations for the lengthening of the bed forms involves the impact of the nonlinearities on the flow field and on sediment transport dynamics, which become significant with an increase in the bed form amplitude and would entail a nonlinear pattern coarsening Linear mechanisms are instead usually considered to be unable to interact with pattern coarsening, and thus, LSA is disregarded for this purpose. As a matter of fact, many experiments have reported that wave lengthening is active soon after the inception of instability, where the core dynamics should not yet be affected by nonlinearities, and a linear analysis is actually acceptable. Moreover, the asymptotic prediction of linear models is well able to fit the equilibrium quantities reported in data. This result has been often questioned as follows: If LSA provides wavelength information for small disturbances, why it fits better long-time than short time observations? We aim of giving a novel answer by making a quantitative comparison between our results and some published experimental data.

As remarked in the Introduction, it is customary to only analyze the least stable eigenvalue of a stability analysis, which only provides information on the asymptotic fate of the disturbance, whereas the ability of the whole spectrum of eigenvalues and eigenfunctions in describing the transient period is usually disregarded. In Sec. IV B, we have showed that a nonmodal LSA is able to detect a transient growth that is active at small wavelengths and which is followed by decay and a long-term increase in the long-wave patterns. In this way, although we are well aware of the presence of nonlinear phenomena which can modify some features of the dynamics (particularly with respect to the time scales), the...
agreement between the theoretical asymptotic predictions of linear analysis and data supports the idea that even—and furthermore—the corresponding linear prediction of transient behavior should be able to capture the main features of the initial lengthening process.

We sustain this idea by comparing the linear transient behavior prediction with some experimental results. To this aim, let us consider the growth function at a fixed time $t$ but still depending on the wavenumber $\alpha$, namely, $G(\alpha) \equiv \hat{G}(\alpha, t)$. This quantity characterizes the energy spectrum of the disturbance at a fixed time. In order to capture the dominant wavenumber, $k(t)$, of the wavenumber spectrum, the first-order moment of the distribution $G(\alpha)$ is accounted for according to $k(t) = (\int_{0}^{\infty} G(\alpha) \, d\alpha)^{-1} \int_{0}^{\infty} \frac{1}{\alpha} G(\alpha) \, d\alpha$. We also introduce the quantity $k_e = \lim_{t \to \infty} k(t)$ as the overall equilibrium wavenumber. In practical terms, $k_e$ is computed at the finite time $t = t_e$, beyond which wavelength variations are no longer appreciable and, coherently to what was stated in Sec. IV A, concerning the limits of the present analysis, the upper bound of the wavenumber was set to $x_{\text{max}} = 10$.

Several growth functions have therefore been computed, varying the wavenumber in the interval $[0, x_{\text{max}}]$ and in time, for three sets of conditions that correspond to three experiments of Coleman et al.,$^{29}$ who investigated the initiation of dunes on a flat sand bed. The results are shown in Figure 7, where the time evolution of the normalized wavenumber $k(t)/k_e$, computed with the present technique, is plotted versus the normalized time $t/t_e$. In agreement with the experiments, the wavenumber undergoes a decrease, i.e., a wave lengthening from the very first instances. Remarkably, the straight lines report the behavior of the power law (21)—in which exponent $\gamma$ is given by the empirical formula by Coleman et al.$^{3}$—which overlaps very well with the behavior of the theoretical analysis in the last part of the evolution ($t/t_e > 0.1$) and approximately in an intermediate part ($0.01 \lesssim t/t_e \lesssim 0.1$ or earlier). These two regions are separated by a short time interval, where a weak peak with a relative maximum is present, beyond which the time evolution returns on the power law decay.

Before the intermediate region, at $t/t_e \lesssim 0.01$, the wavenumber decay is faster than the power law (21), even though the straight line again meets the points at very low times, when the wavenumber is still significantly larger than the equilibrium value. The measurements of the initial wavelengths in the Coleman’s experiments belong to this last wavenumber magnitude and are marked as an arrow on the vertical axis of the plots. It is also worth noticing that, although the upper bound, $x_{\text{max}}$, is imposed, the method is able to capture the so-called “seed waves”$^{1}$ or “wavelets,”$^{29}$ detected by the experiments at very early times. These seed waves are characterized by negative growth factors ($\text{Im}(\lambda_i) < 0$), but transiently positive growth functions ($G_{\alpha}(\alpha) > 0$). These waves are, therefore, triggered by a purely linear non-normal mechanism, which then evolves into a long-wave selection.

Let us here provide a possible physical interpretation of the above results. By keeping all the temporal derivatives in the mathematical model, it has been possible to transfer the total energy among its three components, related to the flow field, free surface, and bed perturbations, respectively. In this way, because of the different timescales and length-scales of the three modes, which are also highlighted by the magnitude of the corresponding eigenvalues, a spatio-temporal evolution of the bed pattern develops. The hydrodynamic modes, in the early period, and the surface mode, in the intermediate period, trigger short-wave instabilities which, because of the non-normal nature of the coupling, also have fallout on the bed dynamics. These disturbances are actually stable at the long-term, thus gradually disappear in the last period, after which the slower long-wave instabilities of the bed emerge. At high Froude numbers, such a scenario is modified by the adjoint of asymptotically unstable roll-waves on the free surface. Nonlinearity plays a marginal role in the earliest period, but becomes relevant in the last period, with a two-fold effect: (1) the amplitude of the disturbances saturates to finite stable values; (2) an energy transfer develops between two or more interacting waves at the same time, thus inducing modulation and deformation of the wave patterns and nonlinear coarse-graining.
At this point, we are able to provide an answer to the afore-mentioned question about the ability of modal LSA in describing dune wavelength selection. During the transitory, shorter disturbances which are stable for the modal analysis (or, at least, less unstable than longer ones) evolve exhibiting transient growths, but eventually the long-term behavior of modal analysis is always recovered, because of long-timescale component of the morphodynamic response. In this context, the presence of two separated timescales in the dynamics, a faster one for the flow and a slower one for the bed, allows any morphodynamic system to behave in opposite manner to the classical hydrodynamic instability; modal analysis is in fact consistently recovered in the long-term behavior.

VI. CONCLUSIONS

Dune patterns are one of the most fascinating phenomena in water and aerial environments and their ubiquity stimulates interest in a multidisciplinary fluid dynamics oriented scientific community. In this paper, we have investigated the non-normal character of the mathematical problem concerning a rotational shear flow model, coupled with sediment and the free surface dynamics. Modal and nonmodal stability analyses have been carried out, and the capability of the system to induce transient growths has been analyzed. Both analyses have been performed while retaining all the time-derivatives.

A refined spectral solution of the eigenvalue problem has allowed us to extract the whole spectrum of eigenvalues and eigenfunctions, and the modal analysis has permitted us to investigate the role of three important modeling aspects concerning the Exner equation, namely, the gravitational, saltation, and inertial corrections. A related aspect has been to recognize that the present model, designed for a rough wall regime, hampers the correct modeling of the very short bed forms (such as ripples). In this respect, an attempt to extend the wavenumber validity of the present theory, through the adjoint of a Van Driest damping function, is expected. Colombini and Stocchino have recently made such an extension adopting a modal approach. The nonmodal one remains to be run.

The nonmodal analysis has been developed in the spirit of a singular value decomposition, where the introduction of a “morphodynamic” energy-norm has allowed us to compute the growth function as well as the numerical abscissa and optimal disturbances. The problem has proved to be remarkably non-normal in most cases and able to develop important transient growths, even for asymptotically stable wavenumbers. This has suggested a possible interpretation of the phenomena of wavelength elongation, where the occurrence of short waves at the inception is replaced by long waves at the long-term. This conjecture was tested by examining some experimental results and appears to be in agreement with a well-established power-law fitting of the wavenumber decay. The herein presented linear mechanism does not demand to replace the effect of nonlinearities. In fact, nonlinearities surely cooperate in carrying the system to the mature condition of the ultimate steady-state. In the long-term period, coalescence occurs (1) when a fast small dune reaches a larger and slower dune runs on its stoss side and then disappears as it reaches the crest and the avalanche face or (2) when a small dune downstream of a larger one is trapped in the recirculation bubble. Therefore, the process of dune formation can be schematized as follows: linear growth of disturbances → linear coarsening up to the wavelengths selected by modal analysis → nonlinear saturation phase up to finite-amplitude (equilibrium amplitude) dunes.

Further improvements in the present analysis would be welcomed, such as the role of tridimensionality of the flow field on the wavelength selection in the spanwise and streamwise directions, the presence of the suspended bedload and the analysis of a time-dependent basic-state through a modified Floquet approach. The latter one has particular relevance on fluvial systems, where the discharge is usually time-dependent and dune dynamics can respond with some delay with respect to the flow field.

ACKNOWLEDGMENTS

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APPENDIX: COMPUTATION OF $\omega_-, \eta, H$

The Exner equation and the kinematic condition at the bottom (Eqs. (11) and (10a), respectively) are summed together—thus eliminating eigenvalue $\lambda$—and inserted into a system with the tangential dynamic condition (8b) and the no slip condition (10b), which in the modal form, provide the following system:

$$w_{-3} + \left[b_{12}\phi^b_2\left(\zeta_b\right) + b_{13}\phi^b_3\left(\zeta_b\right)\right]w_x + b_{14}\eta + b_{15}h = 0,$$

(A1)

$$b_8 w_x \phi^b_3\left(1\right) + b_8 w_{-1} + b_9 \eta + b_{10} h = 0,$$  

(A2)

$$2i w_{-2} + e_{11} \eta = 0.$$  

(A3)

The above linear system does not contain the eigenvalue. Thus, it can be solved apart, providing the expression of $\omega_-, \eta,$ and $h$ as a linear combination of $w_k$ with $k = -3, -1, 0, \ldots, N-3,$ which constitute the components of vector $\mathbf{x}$ and are the ultimate $N$ unknowns of the generalized eigenvalue problem $A\mathbf{x} = \lambda B\mathbf{x}.$


27. See supplementary material at http://dx.doi.org/10.1063/1.3644673 for coefficients a, and b, and a formal derivation of the algebraic eigenvalue problem through a spectral Galerkin method.